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# DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20084



FINITE-ELEMENT METHOD FOR HEAT TRANSFER  
PROBLEM IN HYDRODYNAMIC LUBRICATION

By

Kwang June Bai

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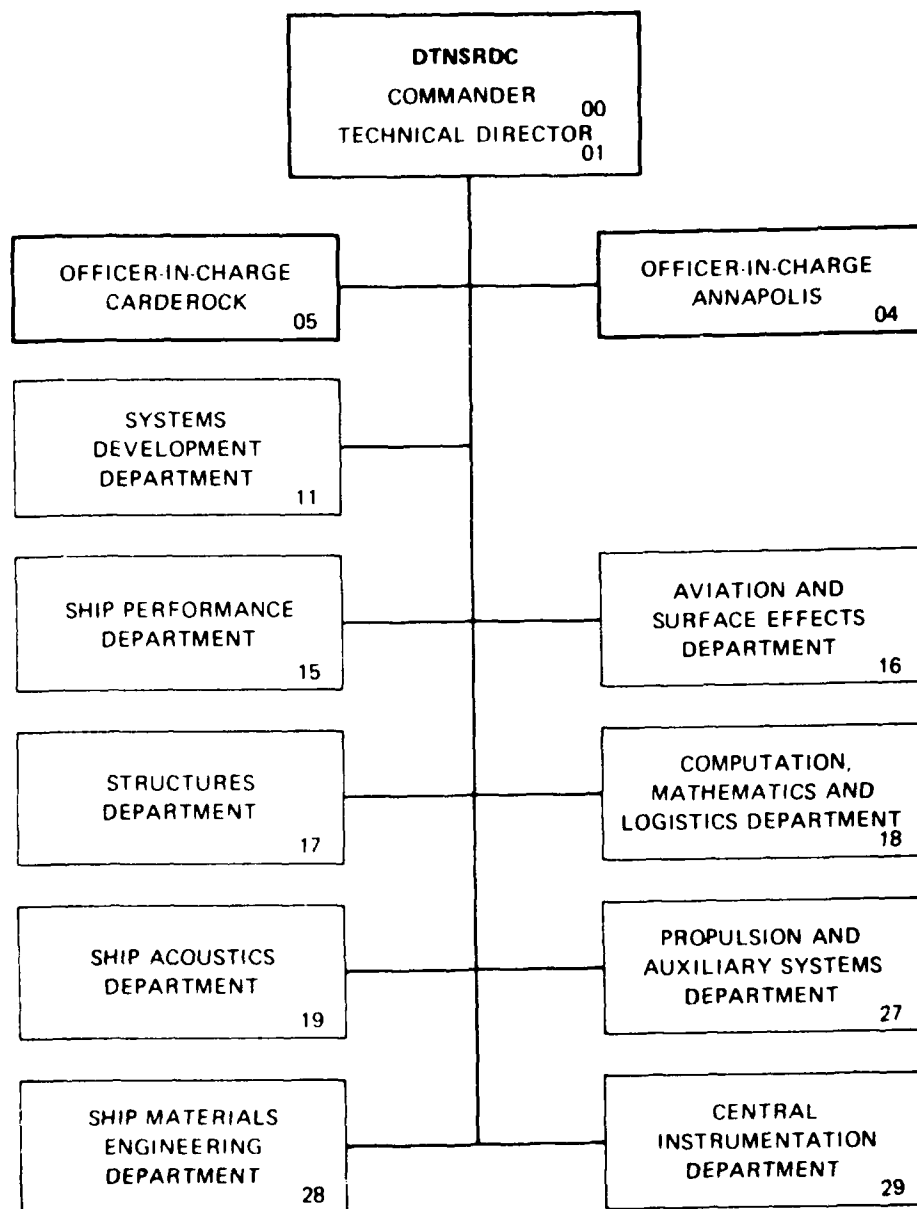
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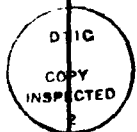
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Block #20 ABSTRACT

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# TABLE OF CONTENTS

	<u>PAGE</u>
LIST OF FIGURES -----	iii
LIST OF TABLES -----	iv
NOTATIONS -----	v
ABSTRACT -----	1
ADMINISTRATIVE INFORMATION -----	1
INTRODUCTION -----	1
MATHEMATICAL MODELING OF THE PROBLEM -----	3
GALERKIN'S METHOD AND NUMERICAL PROCEDURE -----	7
NUMERICAL CONVERGENCE TEST IN THE FIRST MODEL -----	9
RESULTS FOR A SECOND MODEL PROBLEM -----	19
CONCLUDING REMARKS -----	25
ACKNOWLEDGMENT -----	26
REFERENCES -----	27

## LIST OF FIGURES

1 - Boundary Configurations -----	4
2 - Error Convergence Test Results for the Robin Condition -----	12
3 - Error Convergence Test Results for the Neumann Condition -----	14
4 - Error Convergence Test Results for the Dirichlet Condition -----	16
5 - Temperature Distributing for Case 1 $H_2 = 0.001$ inch, $u_0 = 9.55$ in/sec -----	21
6 - Temperature Distributions for Case 2 $H_2 = 0.001$ inch, $u_0 = 47.75$ in/sec -----	22

	<u>PAGE</u>
7 - Temperature Distributions for Case 3 H <sub>2</sub> = 0.0001 inch, u <sub>o</sub> = 9.55 in/sec -----	23
8 - Temperature Distributions for Case 4 H <sub>2</sub> = 0.0001 inch, u <sub>o</sub> = 47.75 in/sec -----	24

#### LIST OF TABLES

1 - Ten Different Finite Element Subdivisions Used for the Error Test -----	10
2 - The Values of n in Various Error Norms for Three Peclet Numbers, P <sub>e</sub> -----	18
3 - The Values of Constant A in Various Error Norms for Three Peclet Numbers, P <sub>e</sub> -----	18
4 - Four Tested Cases for the Values of $\gamma$ and $\Phi$ -----	20

# NOTATIONS

A	Constant in Error Representation
C	Heat Capacity
$E_i$	Error Norm Defined in Equation (13); ( $i = 1, \dots, 4$ )
$f_i$	Known Function on $\Gamma_i$ ( $i = 1$ and $3$ ) in the Neumann Condition
$h_i$	Heat Film Coefficient on $\Gamma_i$ ( $i = 1$ and $3$ ) in the Robin Condition
J	Heat-Mechanical Energy Conversion Factor (= 9336 in lbf/Btu)
$J_i$	Juncture Boundary ( $i = 1, 2$ )
$k_i$	Thermal Conductivity in $\Omega_i$ ( $i = 1, 2, 3$ )
L	Length of Bearing Pad
n	Index of Error
(Oxy)	Rectangular Coordinate Defined in Figure 1
$Pe$	Peclet Number
T	Temperature in Fahrenheit
$T_i$	Temperature in Subdomain $\Omega_i$ ( $i = 1, 2, 3$ )
$T^*$	Test Function in Weak Formulation
$\hat{T}$	Approximate Solution for Temperature
$\vec{U}$	Velocity Vector
(u,v)	Velocity Component in the x and y-axis
$\gamma$	Coefficient of Convective Term
$\Gamma_i$	Boundary of Subdomain $\Omega_i$ ( $i = 1$ and $3$ )
$\Gamma_{2u}$	Inlet (Upstream) Boundary of $\Omega_2$
$\Gamma_{2D}$	Outlet (Downstream) Boundary of $\Omega_2$
$\rho$	Density of Lubricant

$\mu$      Dynamic Viscosity

$\Omega_i$    Three Subdomains (  $i = 1,2,3$  )

#### ABSTRACT

Galerkin's finite element method is applied to a two-dimensional heat convection-diffusion problem arising in the hydrodynamic lubrication of thrust bearings used in naval vessels. A parabolized thermal energy equation for the lubricant, and thermal diffusion equations for both bearing pad and the collar are treated together, with proper juncture conditions on the interface boundaries. It has been known that a numerical instability arises when the classical Galerkin's method, which is equivalent to a centered difference approximation, is applied to a parabolic-type partial differential equation. Probably the simplest remedy for this instability is to use a one-sided finite difference formula for the first derivative term in the finite difference method. However, in the present coupled heat convection-diffusion problem in which the governing equation is parabolized in a subdomain (lubricant), uniformly stable numerical solutions for a wide range of the Peclet number are obtained in the numerical test based on Galerkin's classical finite element method. In the present numerical computations, numerical convergence errors in several error norms are presented in the first model problem. Additional numerical results for a more realistic bearing lubrication problem are presented for a second numerical model.

#### ADMINISTRATIVE INFORMATION

The work reported here was in support of the Tribology Program, which is an interdepartmental effort, under the David W. Taylor Naval Ship Research and Development Center's (DTNSRDC) Independent Research Program Task Area ZR-000-01-01, Work Unit 2832-121-50.

#### INTRODUCTION

The main objective of the present DTNSRDC on-going research in the Tribology Program at DTNSRDC is to develop a reliable tool to predict the behavior of thrust

bearings used in naval vessels over a wide range of operating conditions, with regard to both hydrodynamic and boundary lubrication. The first step toward this goal is to improve the hydrodynamic lubrication prediction. In this first-step approach, an immediate improvement over previous investigations is a full coupling of the thermal energy equation in the lubricant and the heat diffusion in the surrounding bearing and the collar. In this approach, a temperature variation is allowed across the lubricant film thickness and proper matching conditions are imposed on the interface boundaries between the lubricant and adjacent solids. In the present analysis, the standard thermal energy equation is parabolized by assuming that the ratio of the diffusion term to the convection term along the flow direction is of small order.

It has been known that for a parabolic-type partial differential equation an instability arises when the classical Galerkin method is applied, since this method is equivalent to the centered difference approximation. Probably the most extensively studied problem of this type is solution of the well-known boundary layer equations. In the boundary layer equations, most often a one-sided finite difference formula is used, which is equivalent to choosing the basis for the test function space to be different from that for the trial function space in the finite element method. There are many reports describing the introduction of various weighting functions in the inner product or the choice of a test function basis different (that is asymmetric) from the trial function basis [1,2,3,4,5,6,7]. The choice of a test function basis different from the trial function basis plays a role in controlling the degree of "upwinding" to maximize accuracy. Recently, extensive studies have been made for general convection-diffusion equations. However, rigorous investigations into the control of the degree of upwinding are limited to a simple one dimensional model problem with constant coefficients. For more general situations, maximization of accuracy by controlling the degree of upwinding is not straightforward. Therefore,

it is desirable to have a simple numerical scheme which is uniformly stable over a wide range of values of the Peclet number.

In this report numerical results are obtained by the classical Galerkin method. The present numerical scheme gives uniformly stable results over a wide range of Peclet number for the parabolized thermal energy equation in the lubricant. The present numerical scheme is equivalent to the centered difference approximations for both the first and second derivative terms in the original differential equation.

#### MATHEMATICAL MODELLING OF THE PROBLEM

The computation domain consists of three rectangular subdomains, having heights  $H_1$ ,  $H_2$ ,  $H_3$ , and length  $L$  as shown in Figure 1. However, the choice of actual boundary geometry is general (not necessarily rectangular domain) in the present method. The upper, middle, and lower subdomains are the bearing pad, lubricant, and the bearing collar, denoted by  $\Omega_1$ ,  $\Omega_2$ , and  $\Omega_3$ , respectively. A rectangular co-ordinate system is used, the y-axis pointing upward and the x-axis pointing toward the right-hand side. The origin is taken at the mid-point of the left-hand side vertical boundary of the lubricant. The boundary of each subdomain is as shown in Figure 1,

$$\begin{aligned} \partial\Omega_1 &= \Gamma_1 \cup J_1 & (i = 1 \text{ and } 3) \\ \partial\Omega_2 &= \Gamma_{2U} \cup \Gamma_{2D} \cup J_1 \cup J_2 & (1) \end{aligned}$$

Here the flow velocity  $\hat{U}$  is also shown in Figure 1.

In the present numerical test, we ignore the convection term in the solids, i.e.,  $\Omega_1$  and  $\Omega_3$ , by assuming they are stationary. The thermal conductivity,  $k_i$  ( $i=1,3$ ) is assumed constant in each subdomain. In the subdomain  $\Omega_2$  (the lubricant), the velocity distribution is specified a priori, hence the heat generating source term

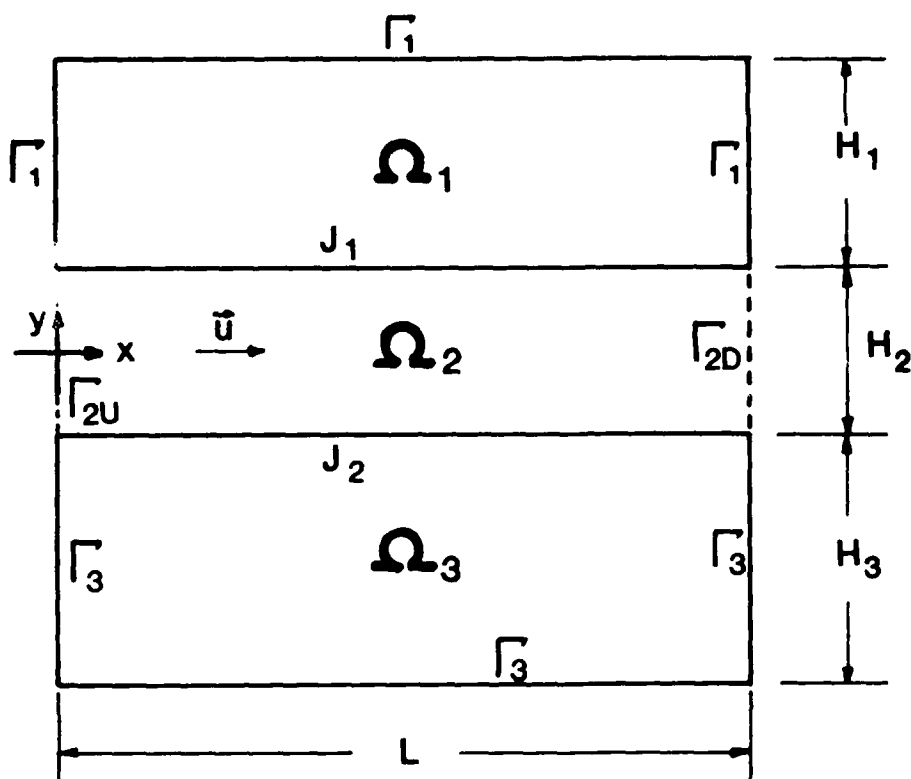


Fig. 1 Boundary Configurations

$\Phi(x, y)$  is known. The density  $\rho$  and the specific heat  $C$  of the lubricant are also assumed to be constants in  $\Omega_2$ . Furthermore we assume that the diffusion term is much smaller than the convection term along the x-axis, and that the convection term is much smaller than the diffusion term in the y axis. From these assumptions the original thermal energy equation, of an elliptic type in the lubricant, can be parabolized as follows. Let the temperature  $T_i (i=1, 2, 3)$  be defined in each corresponding subdomain  $\Omega_i$ . Then the thermal energy equation in each subdomain can be written as

$$\begin{aligned}
-k_i \nabla^2 T_i &= 0 & \text{in } \Omega_i, (i = 1 \text{ and } 3) \\
-k_2 \frac{\partial^2 T_2}{\partial y^2} + \gamma \frac{\partial T}{\partial x} &= \Phi(x, y) & \text{in } \Omega_2
\end{aligned} \tag{2}$$

where  $\gamma = C\rho u$ , and  $\Phi = \mu(\frac{\partial u}{\partial y})^2$  and  $\mu$  is the dynamic viscosity. As mentioned before we assumed that the term  $|k_2 \frac{\partial^2 T_2}{\partial x^2} / C\rho u \frac{\partial T}{\partial x}| = o(1)$  and  $|C\rho v \frac{\partial T}{\partial y} / k_2 \frac{\partial^2 T_2}{\partial y^2}| = o(1)$  in the thermal energy equation in  $\Omega_2$ , where  $u$  and  $v$  are, the velocity components along the  $x$ -axis and  $y$ -axis, respectively.

Since the thermal energy equation in  $\Omega_2$  is parabolized, we impose the boundary condition only on the upstream boundary,  $\Gamma_{2U}$ , and proper juncture conditions on  $J_1$  and  $J_2$ , i.e.,

$$T_2 = T_o \quad \text{on } \Gamma_{2U} \tag{3}$$

where the temperature of incoming lubricant,  $T_o$ , is specified. At the juncture surfaces  $J_1$  and  $J_2$  we require continuity of the temperature and its normal heat flux, i.e.,

$$T_1 = T_2 \quad \text{and} \quad k_1 T_{1y} = k_2 T_{2y} \quad \text{on } J_1 \tag{4}$$

and

$$T_2 = T_3 \quad \text{and} \quad k_2 T_{2y} = k_3 T_{3y} \quad \text{on } J_2$$

In the first numerical model problem, the following three standard types of boundary conditions on  $\Gamma_1$  and  $\Gamma_3$  are treated specifically for testing numerical convergence:

$$(i) \text{ Dirichlet type; } T_i = T_o \quad \text{on } \Gamma_i \quad (i = 1 \text{ and } 3) \tag{5}$$

where  $T_o$  is specified.

$$(ii) \text{ Neumann type; } k_i \frac{\partial T_i}{\partial n} = f_i(x, y) \quad \text{on } \Gamma_i \quad (i = 1 \text{ and } 3) \quad (6)$$

where  $f_i$  ( $i=1$  and  $3$ ) is specified.

$$(iii) \text{ Robin type; } k_i \frac{\partial T_i}{\partial n} + h_i T_i = h_i T_o \quad \text{on } \Gamma_i \quad (i = 1 \text{ and } 3) \quad (7)$$

where the heat transfer film coefficient  $h_i$  ( $i=1$  and  $3$ ) and the ambient oil temperature distribution  $T_o(x, y)$  along  $\Gamma_1$  and  $\Gamma_3$  are specified. Here we assume the boundaries,  $\Gamma_1$  and  $\Gamma_3$  are in an oil bath with an ambient oil temperature  $T_o(x, y)$ . It should be noted here that the boundary condition on the downstream boundary (outlet) of lubricant,  $\Gamma_{2D}$ , is not specified but is obtained as a part of the solution. This is because the original elliptic equation has been reduced to a parabolic equation.

For the purpose of the error and convergence test in the first model problem, we take  $H_1 = H_2 = H_3 = 2$ ,  $k_1/k_2 = 3$ ,  $k_1 = k_3 = 3$ ,  $h_1 = h_3 = 1$ , and the juncture boundaries  $J_1$  and  $J_2$  are  $y = \pm 1$ , respectively. We begin with the exact solution given by a simple polynomial function in each subdomain as follows;

$$\begin{aligned} T_1(x, y) &= -2x^2 + x + 2y^2 - 3y + 3 & \text{in } \Omega_1 \\ T_2(x, y) &= -2x^2 + x + y^2 + y & \text{in } \Omega_2 \\ T_3(x, y) &= -2x^2 + x + 2y^2 + \frac{11}{3}y + \frac{5}{3} & \text{in } \Omega_3 \end{aligned} \quad (8)$$

From a given arbitrarily specified function of  $y$ , one can easily compute the heat generation  $\Phi$  by Eqs (2) and (8) as

$$\Phi(x, y) = y(4x - 1) + 2 \quad (9)$$

The three different types of boundary conditions in this model problem are computed from the known exact solutions given in Eq(8). For example, with the Robin type condition,  $h_i = 1$  ( $i=1,3$ ),  $T_0(x,y)$  was computed from Eq (8) as

$$T_0 = \frac{k_i}{h_i} \frac{\partial T_i}{\partial n} + T_i \quad \text{on } \Gamma_i, \quad (i = 1 \text{ and } 3)$$

and

$$T_0 = \gamma^2 + \gamma \quad \text{on } \Gamma_{2U} \quad (10)$$

In the second model problem, we only treat the Robin boundary condition on  $\Gamma_1$  and  $\Gamma_3$  to simulate a more realistic experimental condition. For this case, the geometry of the bearing pad and collar and the lubricant film thickness are chosen as an analogous model in two dimensions corresponding to the three-dimensional experimental condition.

#### GALERKIN'S METHOD AND NUMERICAL PROCEDURE

Before we describe Galerkin's finite element method applied to the model problem formulated in the previous section, it is convenient to introduce a single continuous temperature function  $T(x,y)$ , defined in the entire domain,  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ , as follows;

$$T(x,y) = T_i(x,y) \quad \text{in } \Omega_i \quad (i=1,2,3). \quad (11)$$

To construct the bilinear functional in weak form, we first introduce the test function  $T^*$  in the test function space, and next define the inner product of the original partial differential equations in (2) and the test function  $T^*$ . By integrating by parts the inner product reduces to

$$\begin{aligned}
& \iint_{\Omega_1} K_1 \nabla T \nabla T^* dx dy + \iint_{\Omega_2} K_2 \frac{\partial T}{\partial y} \frac{\partial T^*}{\partial y} dx dy + \iint_{\Omega_3} K_3 \nabla T \nabla T^* dx dy \\
& + \int_{\Gamma_1} h_1 T T^* ds + \int_{\Gamma_3} h_3 T T^* ds + \iint_{\Omega_1} f \frac{\partial T}{\partial x} T^* dx dy = \iint_{\Omega_2} \Phi T^* dx dy \quad (12) \\
& + \int_{\Gamma_1} h_1 T_0 T^* ds + \int_{\Gamma_3} h_3 T_0 T^* ds
\end{aligned}$$

where the trial function  $T^* = 0$  on  $\Gamma_{2U}$ , and the trial function  $T$  is chosen so that the essential condition  $T = T_0$  on  $\Gamma_{2U}$  is satisfied. Eq (12) is a weak form for the Robin condition. For the Dirichlet type condition on  $\Gamma_1$  and  $\Gamma_3$ , the line integrals along  $\Gamma_1$  and  $\Gamma_3$  in Eq (12) are not present. On the other hand the trial function should satisfy the Dirichlet conditions and the test function is chosen to be zero on  $\Gamma_1$  and  $\Gamma_3$ . In the case of the Neumann type condition, the boundary integrals along  $\Gamma_1$  and  $\Gamma_3$  appearing on the left-hand side of Eq (12) should not be present and  $h T_0$  in the integrands of the boundary integrals along  $\Gamma_1$  and  $\Gamma_3$ , on the right-hand side of Eq (12), should be replaced by  $f_i(x,y)$ , ( $i=1$  and  $3$ ).

It is of interest to note that the juncture conditions on  $J_1$  and  $J_2$  given in Eq (4) are satisfied as natural conditions in Galerkin's functional equation given above. In the numerical computations an isoparametric linear element is used as the basis for both trial and test functions throughout the present computations. This choice of basis function is equivalent to the centered finite difference approximation. In a straightforward manner, the bilinear form in Eq (12) is reduced to a set of algebraic equations. The coefficient matrix obtained is not symmetric but still has a banded structure. The asymmetry is due to the presence of the

convective term in a subdomain . The Gaussian elimination method is used to solve the reduced matrix equation.

#### NUMERICAL CONVERGENCE TEST IN THE FIRST MODEL

An extensive numerical test of the convergence has been made in the first model problem. To test numerical convergence of the present numerical scheme, we define the error,  $E_i$  ( $i=1,4$ ), in four different ways using the known exact solution  $T$  given in Eq (8) and the finite element numerical solution  $\hat{T}(x,y)$  as follows:

$$\begin{aligned} E_1 &= \|T - \hat{T}\|_{\infty} \\ E_2 &= \left\{ \iint (T - \hat{T})^2 dx dy + \iint k [\nabla(T - \hat{T})]^2 dx dy \right\}^{1/2} \\ E_3 &= \|\nabla(T - \hat{T})\|_2 \\ E_4 &= \|T - \hat{T}\|_2 \end{aligned} \tag{13}$$

where  $\|\cdot\|_{\infty}$  and  $\|\cdot\|_2$  are the well-known max norm and  $L_2$  norm, and defined as

$$\begin{aligned} \|T - \hat{T}\|_{\infty} &= \max_{(x,y) \in \Omega} |T - \hat{T}| \\ \|T - \hat{T}\|_2 &= \left\{ \iint (T - \hat{T})^2 dx dy \right\}^{1/2} \end{aligned} \tag{14}$$

and where  $k = k_i$  in  $\Omega_i$  ( $i=1,2,3$ )

In the finite element mesh subdivisions,  $\Delta x / \Delta y = 1$  is used throughout the first model problem, where  $\Delta x$ , and  $\Delta y$  are the lengths of the finite element along the  $x$ - and  $y$ - axes respectively (i.e., a square element is used for this model problem). Ten different sizes of uniform square elements are tested in the range of

$2 \leq L/\Delta x \leq 20$  . The specific mesh-subdivisions tested are given in

Table 1. In the present model problem, the computations are made for three values of Peclet number:  $Pe = \gamma H_2/2 = 0.01, 1, \text{ and } 100$ .

Table 1 Ten different finite element subdivisions used for the error test.

Test case (1)	1	2	3	4	5	6	7	8	9	10
I	2	4	6	8	10	12	14	16	18	20
J	3	6	9	12	15	18	21	24	27	30
EL	6	24	54	96	150	216	294	384	486	600
N	12	35	70	117	176	247	330	425	532	651
MB	9	17	23	29	35	41	47	53	59	65
DIM	108	595	1610	3395	6160	10127	15510	22525	31388	42315

I = Number of element along the x-axis

J = Number of element along the y-axis

EL = Total number of elements

N = Total number of nodes

MB = Bandwidth (asymmetric)

DIM = Core memory space for the coefficient matrix

If we assume that the error behaves like  $E_i \propto (\Delta x)^n$  ( $i=1,2,3,4$ ) as the limit  $\Delta x \rightarrow 0$ , then we may represent the error as

$$E_i = A (1/I)^n, \quad (15)$$

where A is constant and I is defined in Table 1. By taking the log of Eq (15), we obtain

$$\ln E_i = \ln A - n \ln I \quad (16)$$

From our numerical results for ten different finite-element mesh subdivisions,

we have plotted the curve of the points  $(-\ln A, \ln E_1)$  shown in Figs. 2 through 4. From the two finest finite element subdivisions, the values of index  $n$  and the constant  $A$  are obtained for three different values of the Peclet number,  $Pe$ , and also for three types of boundary conditions on  $\Gamma_1$  and  $\Gamma_3$ . The results are given in Table 2 and 3. In Table 2 the values of  $n$  for  $E_2$  and  $E_3$  are almost one, i.e., the corresponding convergence error is linear, as a function of  $(1)^{-1}$  for all three types of boundary conditions and all three Peclet numbers tested except for  $Pe=100$  in the Neumann boundary condition. It is surprising to see that the convergence of the  $E_1$  error is accelerated as the Peclet number increases - this is contrary to our expectation. It is also difficult to draw any conclusion on the behavior on the convergence of the error  $E_4$  as the Peclet number increases.

In Table 3, the constant  $A$  for  $E_1$  increases as the Peclet number increases with all three types of boundary conditions on  $\Gamma_1$  and  $\Gamma_3$ . However, the constant  $A$  for  $E_2$  and  $E_3$  does not vary much for all three types of boundary conditions and for the three values of Peclet numbers.

From the results shown in Figure 3 and Table 2 and 3, the present numerical evidence shows uniform convergence of the present numerical scheme. However, a rigorous mathematical error analysis and convergence proof is still open for the class of parabolic-elliptic coupled problem treated here.

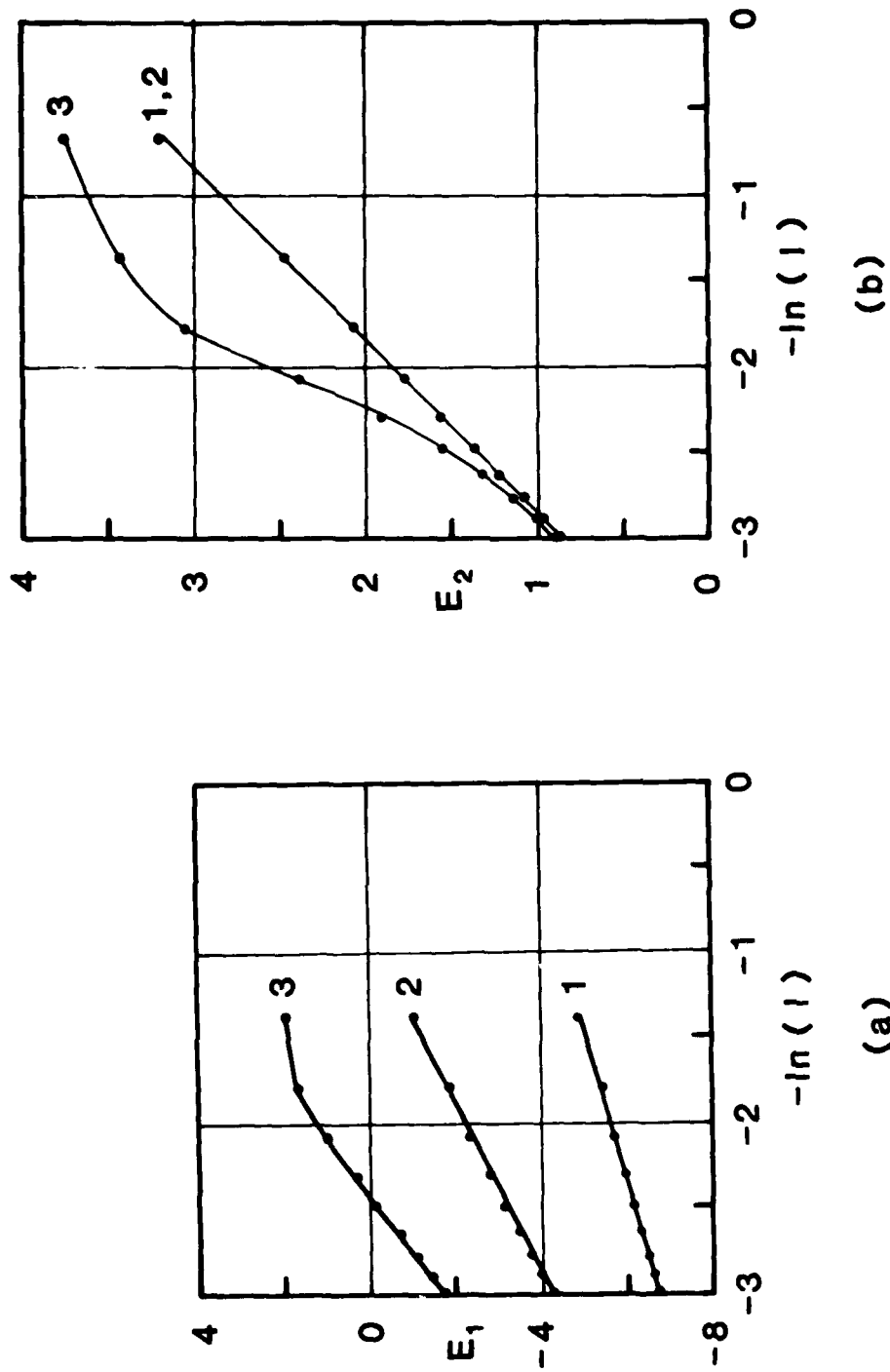


Figure 2. Error convergence test results for the Robin condition.  
 In Figure 2 through Figure 4, the following legends are used:  
 (a), (b), (c), (d) are the results of  $E_1$ ,  $E_2$ ,  $E_3$ , and  $E_4$ , respectively (see Eq 13)  
 The ordinate scale is the same in all figures. The curves designated by  
 1, 2, and 3 correspond to the case of  $p_e = 0.01$ , 1, and 100, respectively.

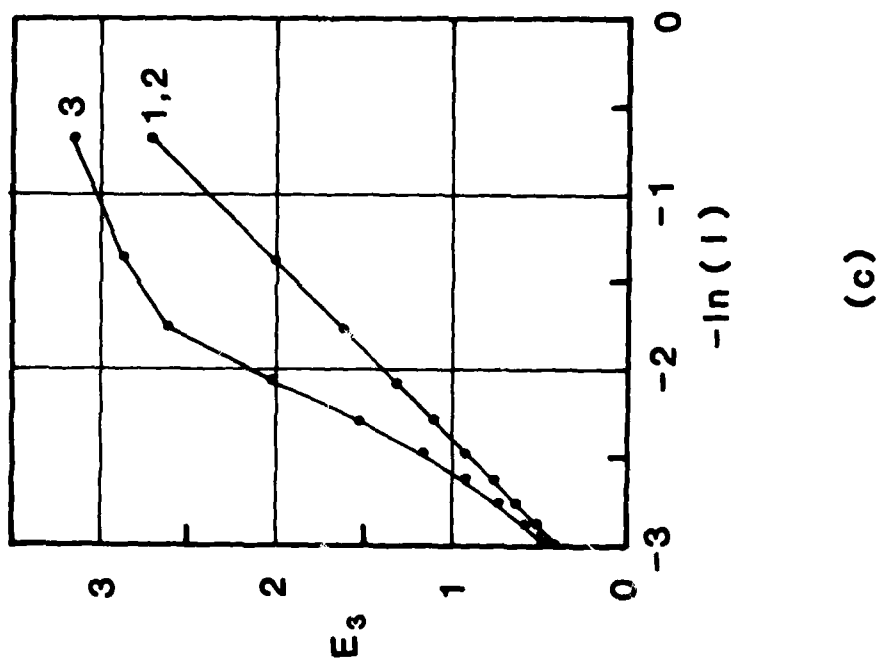
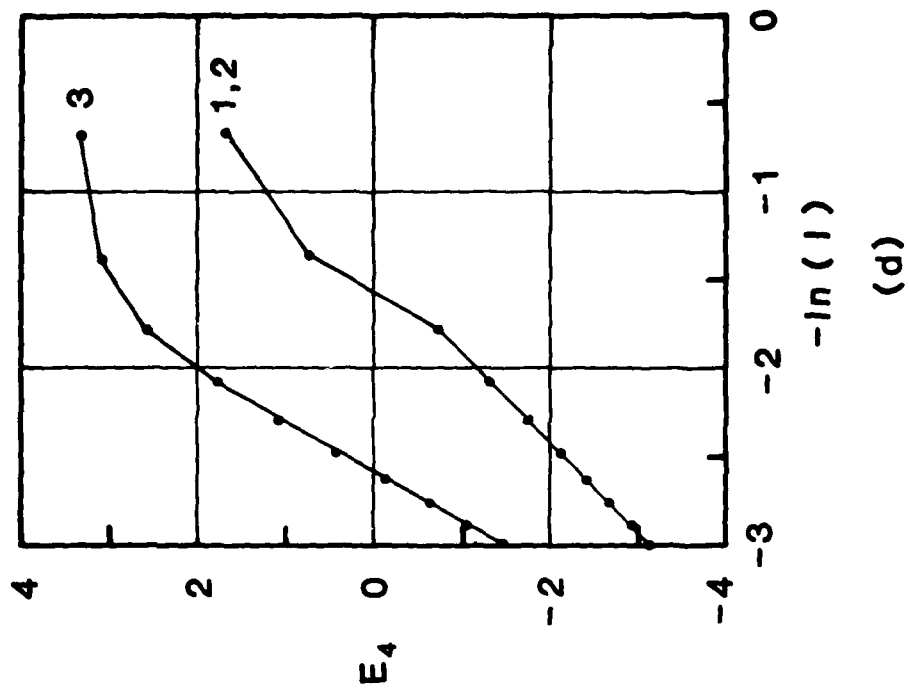
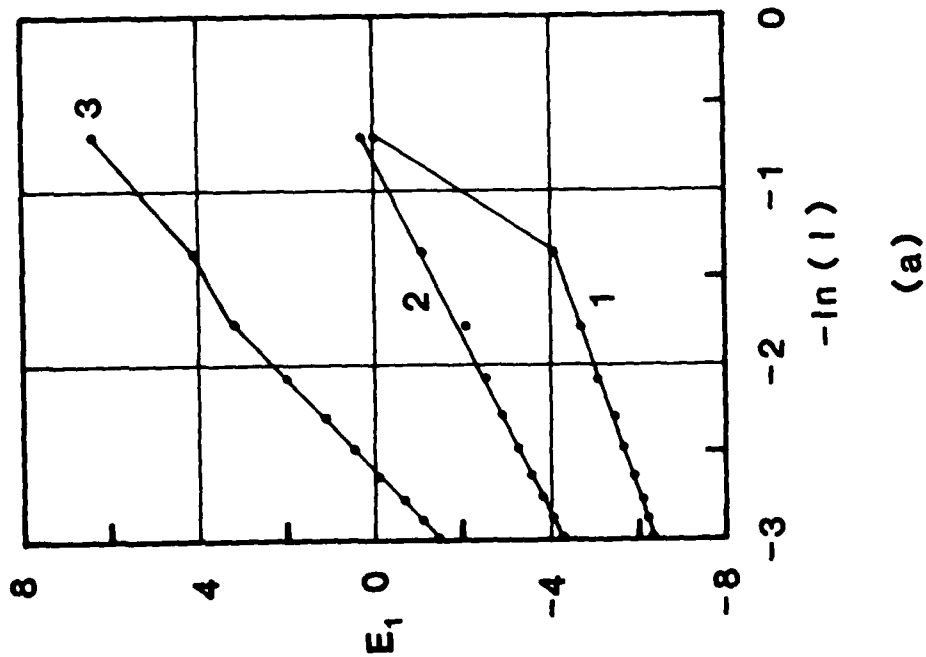
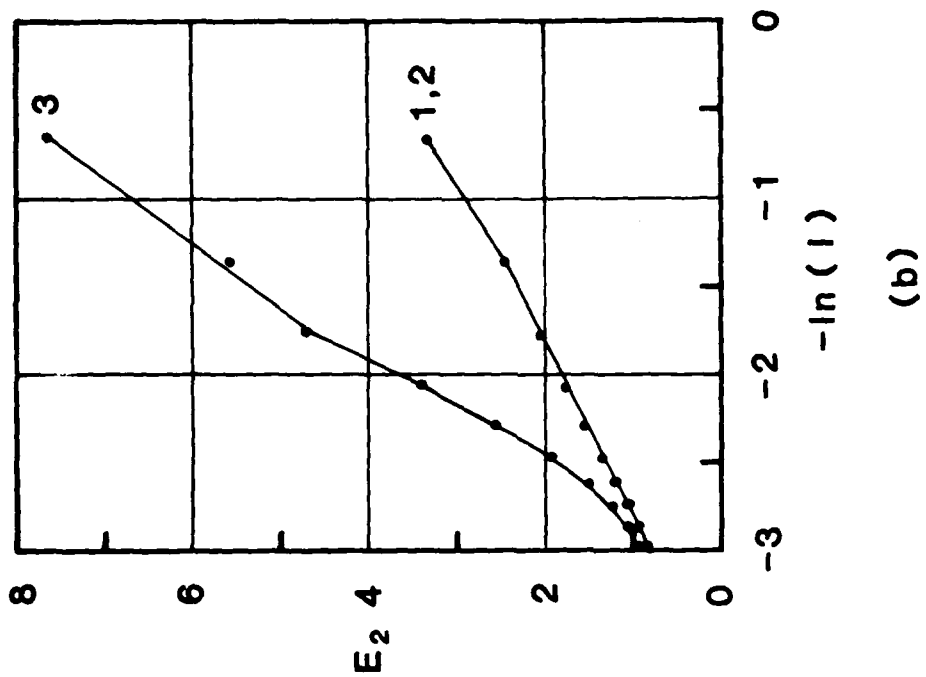


Figure 2 (Continued)



(a)



(b)

Figure 3 Error convergence test results for the Neumann condition.  
(See Figure 2 for the legend)

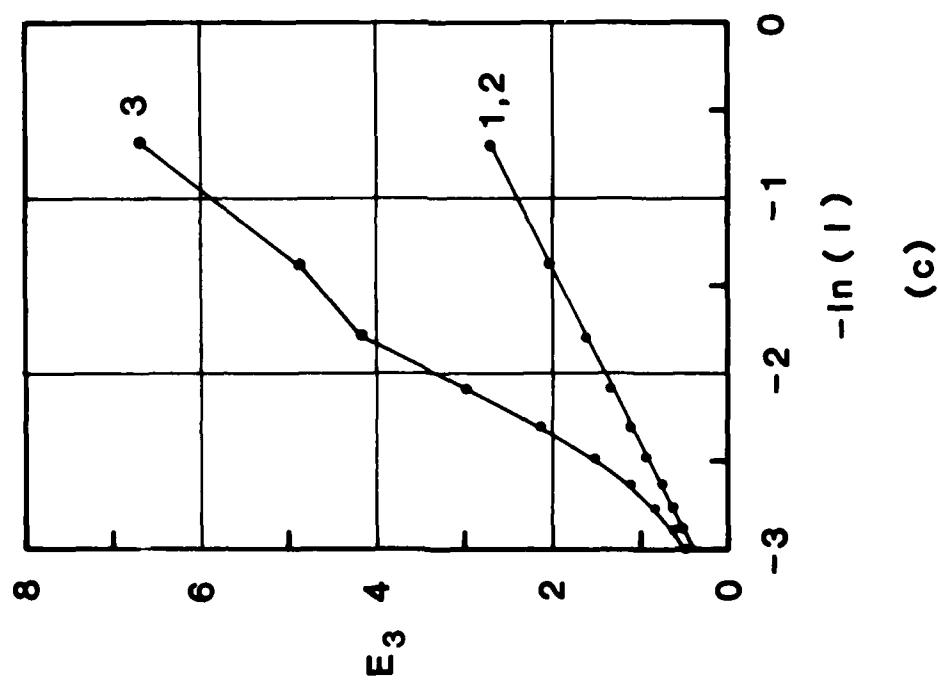
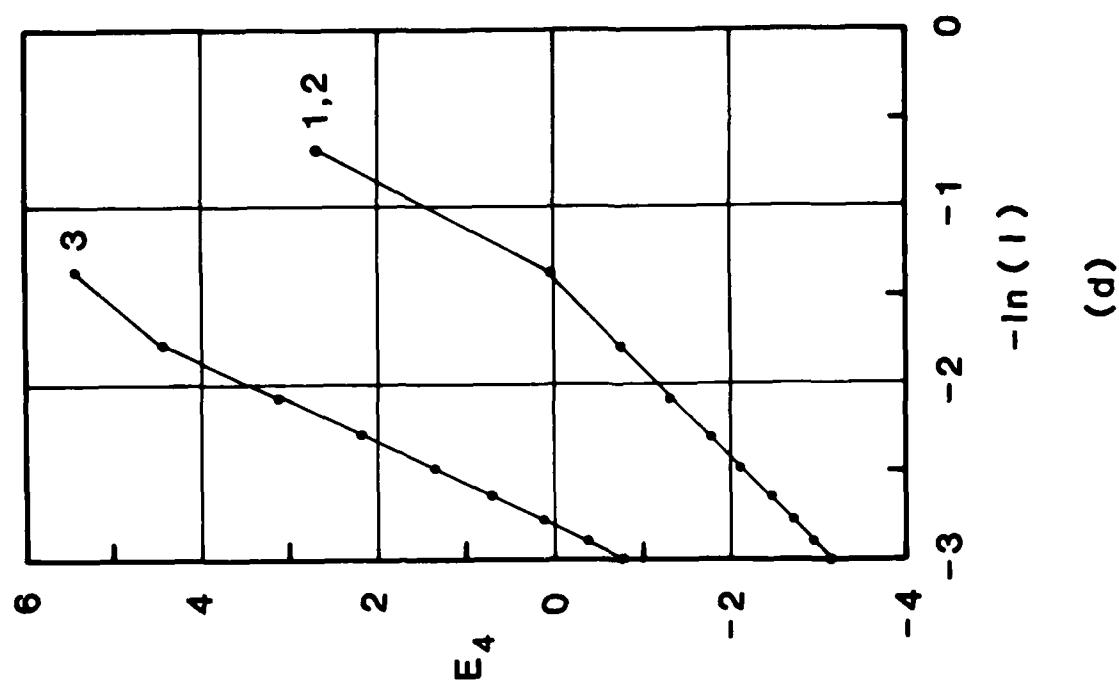


Figure 3 (Continued)

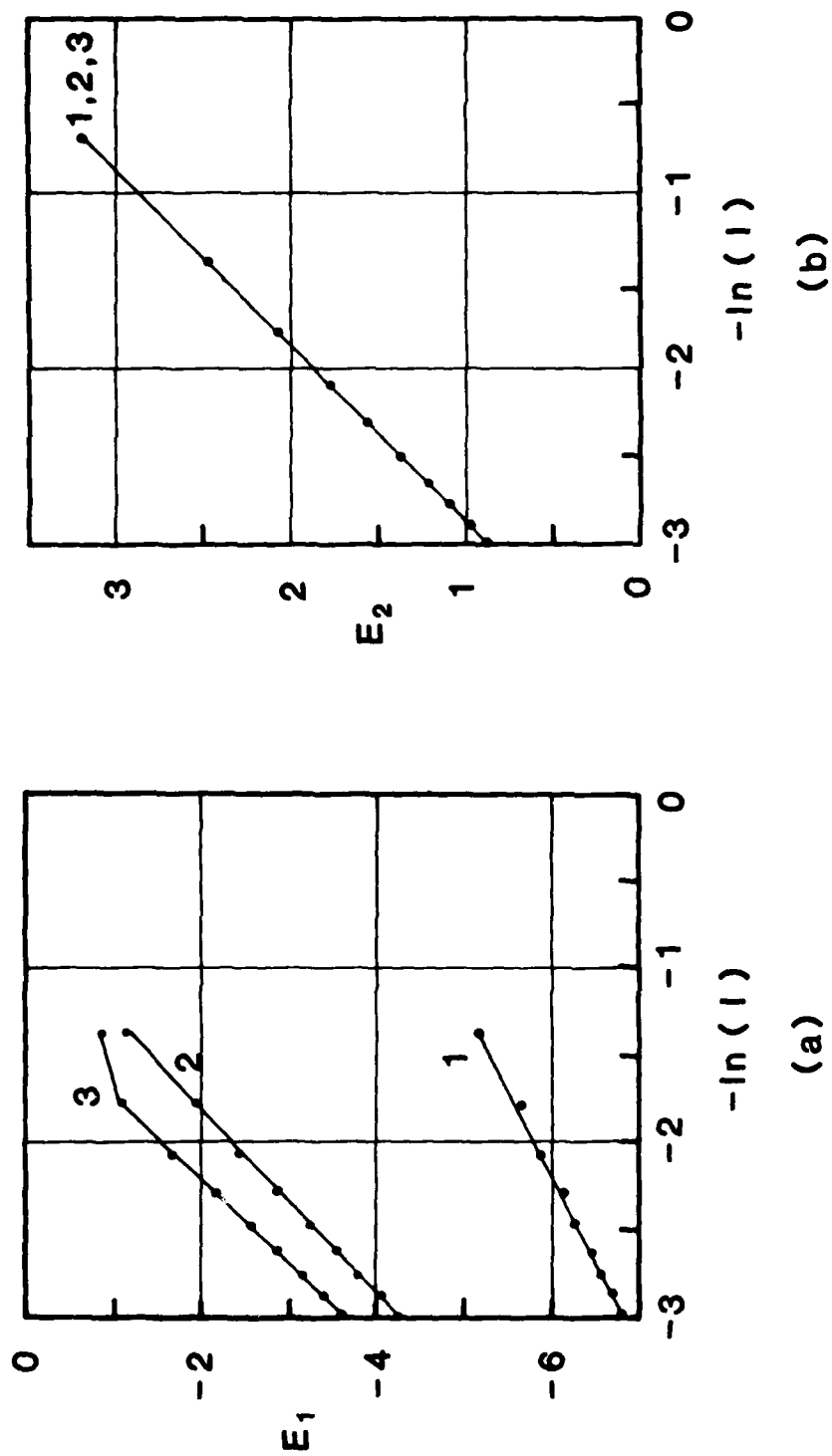


Figure 4 Error convergence test results for the Dirichlet condition.  
(See Figure 2 for legend)

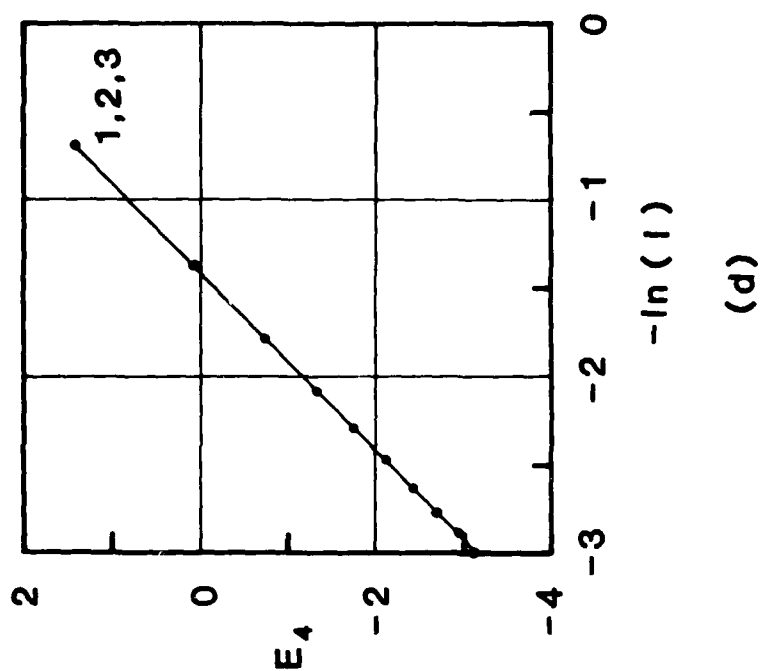
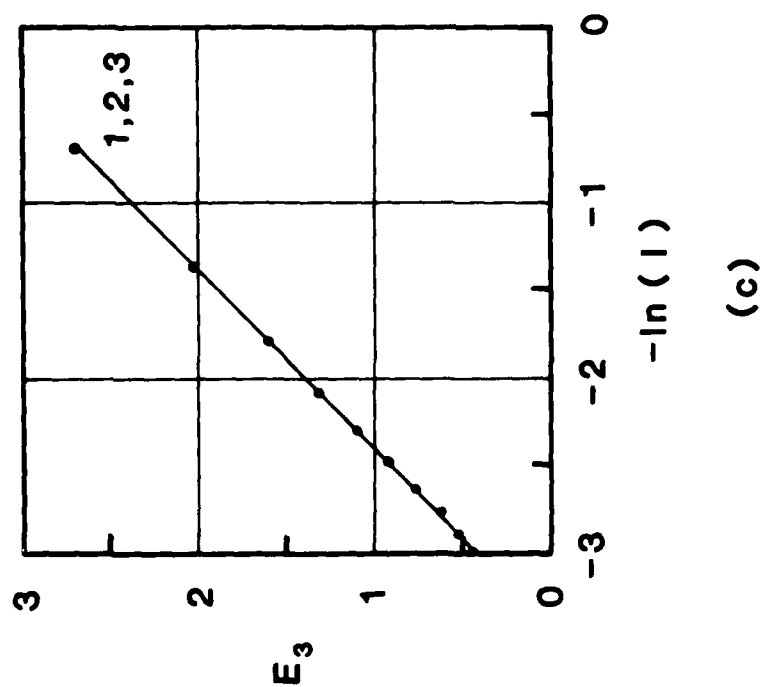


Figure 4 (Continued)

Table 2. The Values of  $n$  in various Error Norms for Three Peclet Numbers,  $P_e$ .

Boundary Condition	$P_e$	$E_1$	$E_2$	$E_3$	$E_4$
Robin	0.01	1.0038	1.0004	1.0000	2.0000
	1	2.0607	1.0004	1.0000	1.9971
	100	3.2114	1.1082	1.1482	3.2223
Neumann	0.01	1.2871	1.0004	1.0000	1.9985
	1	2.0098	1.0004	1.0000	1.9834
	100	3.6481	1.3138	1.3709	4.1813
Dirichlet	0.01	.9175	1.0004	1.0000	2.0000
	1	2.0366	1.0003	.9999	2.0003
	100	2.1011	1.0009	1.0008	1.9957

Table 3. The Values of Constant  $A$  in various Error Norms for Three Peclet Numbers,  $P_e$ .

Boundary Condition	$P_e$	$E_1$	$E_2$	$E_3$	$E_4$
Robin	0.01	.2481E-01	.4761E+02	.2993E+02	.1721E+02
	1	.7006E+01	.4769E+02	.3005E+02	.1726E+02
	100	.2580E+04	.6760E+02	.4908E+02	.1601E+05
Neumann	0.01	.8260E-01	.4761E+02	.2993E+02	.1740E+02
	1	.5876E+01	.4769E+02	.3005E+02	.1641E+02
	100	.1299E+05	.1281E+03	.9813E+02	.1257E+06
Dirichlet	0.01	.1740E-01	.4761E+02	.2993E+02	.1720E+02
	1	.6339E+01	.4768E+02	.3004E+02	.1747E+02
	100	.1472E+02	.4823E+02	.3085E+02	.1806E+02

## RESULTS FOR A SECOND MODEL PROBLEM

For the second model problem, the following geometrical and material data are used:

$$H_1 = H_3 = 0.75 \text{ inch}$$

$$H_2 = 0.001 \text{ and } 0.0001 \text{ inch}$$

$$L = 2.5 \text{ inch}$$

$$u_o = 9.55 \text{ inch/sec and } 47.75 \text{ inch/sec}$$

$$C = 0.5 \text{ Btu/lbm/}^\circ\text{F}$$

$$k_1 = k_3 = 26 \text{ Btu/hr/ft/}^\circ\text{F}$$

$$k_2 = 0.075 \text{ Btu/hr/ft/}^\circ\text{F}$$

$$T_o = 100 \text{ }^\circ\text{F}$$

$$h_1 = h_3 = 30 \text{ Btu/hr/ft}^2\text{/}^\circ\text{F}$$

$$\mu = 6.5 \times 10^{-4} \text{ lbm.sec/in}$$

$$\rho = 0.84 \times 10^{-3} \text{ lbf sec /in}$$

$$J = 9336 \text{ in.lbf/Btu}$$

where  $u_o$  is the maximum velocity in the lubricant film. The value of  $\gamma$  is approximated by

$$\Phi(x, y) = \mu \left( \frac{\partial u}{\partial y} \right)^2 \approx \mu \left( \frac{u_o}{H_2} \right)^2$$

and  $\gamma$  is computed by using the mean velocity, i.e.,  $u = u_o/2$ , since the velocity is assumed linear between zero on  $J_1$  and  $u_o$  on  $J_2$ .

If all the dimensional quantities are converted to consistent dimensions using (Btu, sec, in,  $^\circ\text{F}$ ), then the following values of coefficients are obtained for use in Eqs (2) and (7):

$$K_1 = K_3 = 602 \times 10^{-6} \frac{\text{Btu}}{\text{sec in}^2 \text{ } ^\circ\text{F/in}}$$

$$K_2 = 1.74 \times 10^{-6} \frac{\text{Btu}}{\text{sec in}^2 \text{ } ^\circ\text{F/in}}$$

$$h_1 = h_3 = 57.87 \times 10^{-6} \frac{\text{Btu}}{\text{sec in}^2 \text{ } ^\circ\text{F}}$$

Table 4 shows four specific test conditions, which are the combinations of two two velocities and two film thickness.

Table 4. Four tested cases for the values of  $\gamma$  and  $\Phi$ .

Case No.	$H_2$ (inches)	$u_o$ (inch/sec)	$\gamma$ (Btu/sec/in / $^\circ\text{F}$ )	$\Phi$ (Btu/in /sec)
1	0.001	9.550	0.0810	0.06427
2	0.001	47.750	0.0405	1.6070
3	0.0001	9.550	0.0810	6.4270
4	0.0001	47.750	0.0405	160.7000

In the present computations, two different sets of mesh subdivisions are used; the first set of data with a coarse mesh has 77 nodes and 60 elements. The second set of data with a fine mesh has 315 nodes and 280 elements. In the fine mesh, we took four uniform rectangular elements in the lubricant ( $\Omega_2$ ) and five uniform rectangular elements in both bearing pad and collar ( $\Omega_1$  and  $\Omega_3$ ) along the y-axis, and twenty elements along the x-axis.

The agreement between the computed temperatures obtained by a fine mesh and a coarse mesh was good. Therefore only the results obtained by using the fine mesh are shown in Fig 5 through 8. In this three dimensional computer plot of the temperature distribution, a total of 315 nodes were used with linear interpolation. It should be noted here that the film thicknesses (i.e., 0.001 and 0.0001 inches) were stretched

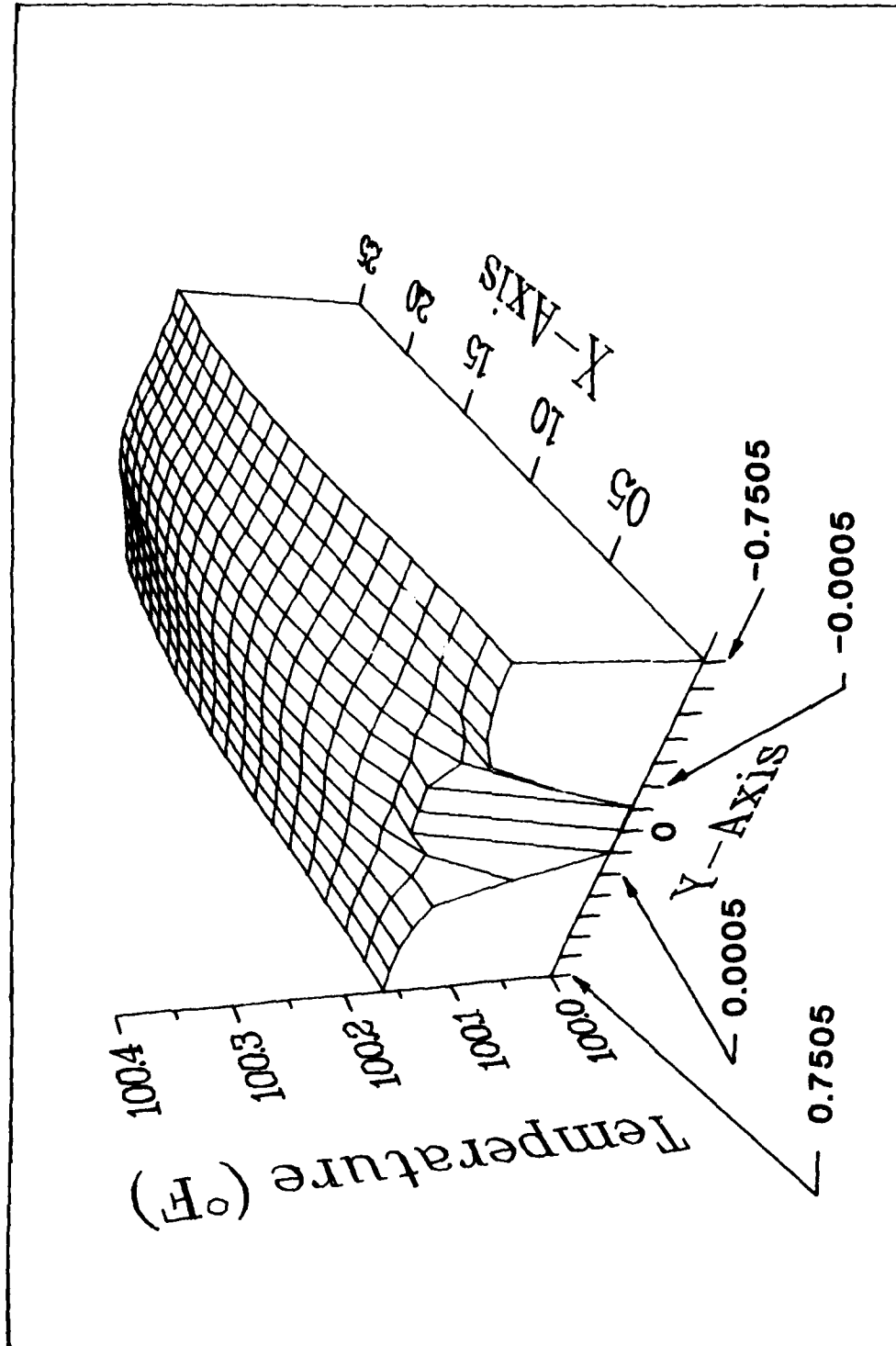


Figure 5 Temperature distributions for Case 1.

$H_2 = 0.001$  inch,  $u_0 = 9.55$  in/sec

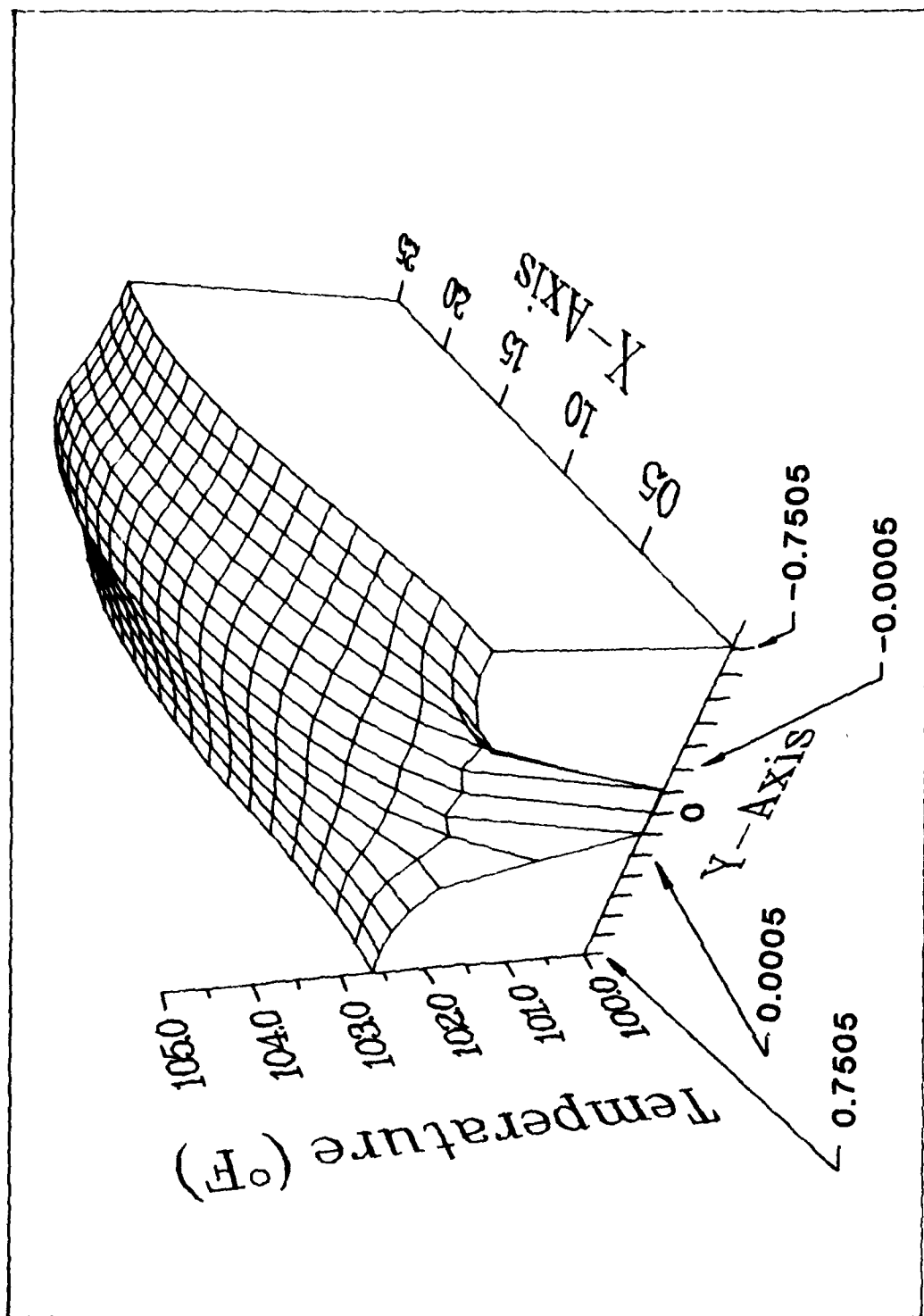


Figure 6. Temperature distributions for Case 2.

$H_2 = 0.001$  inch,  $u_0 = 47.75$  in/sec

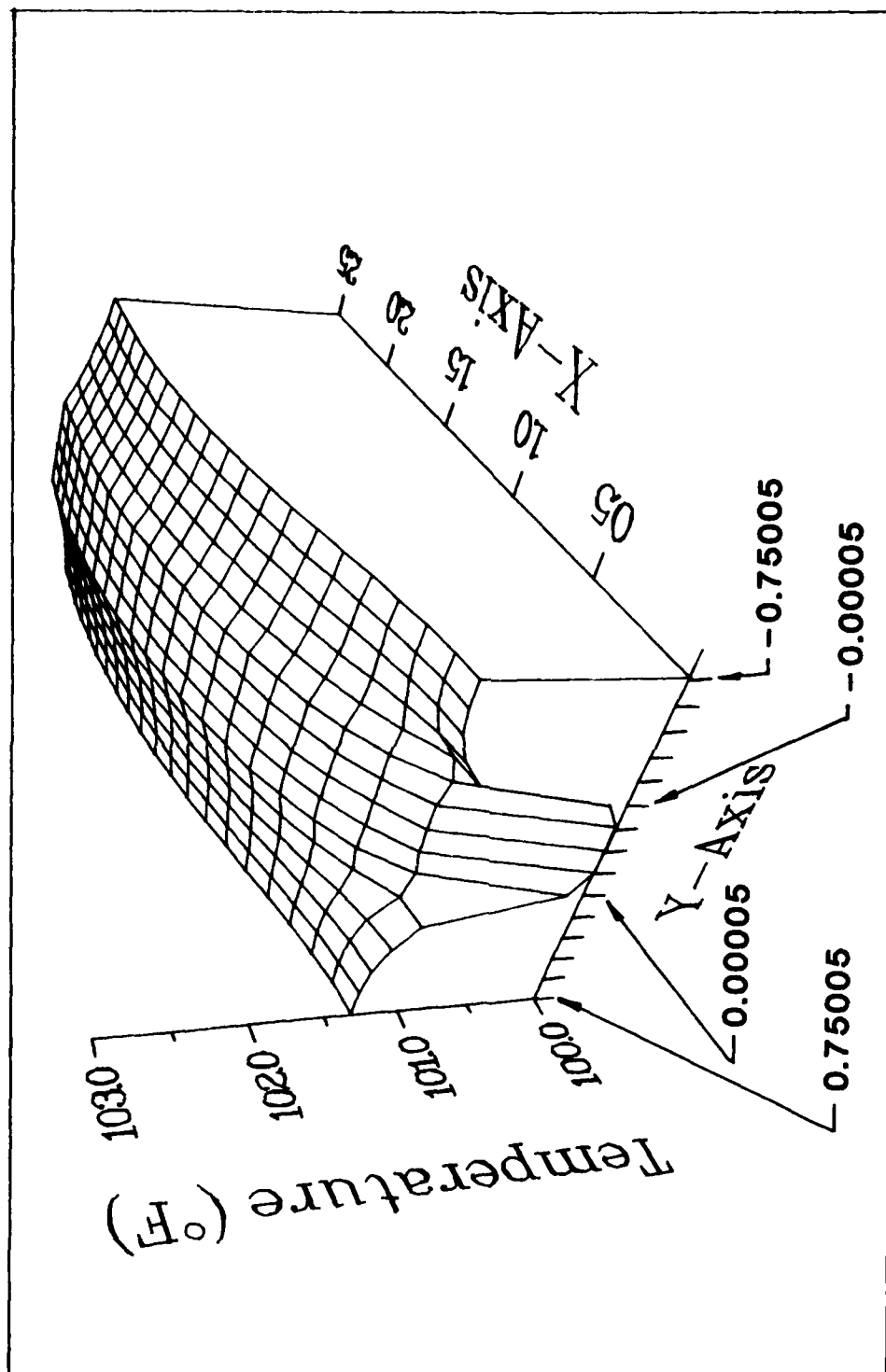


Figure 7. Temperature distributions for Case 3  
 $H_2 = 0.0001$  inch,  $u_0 = 9.55$  in/sec

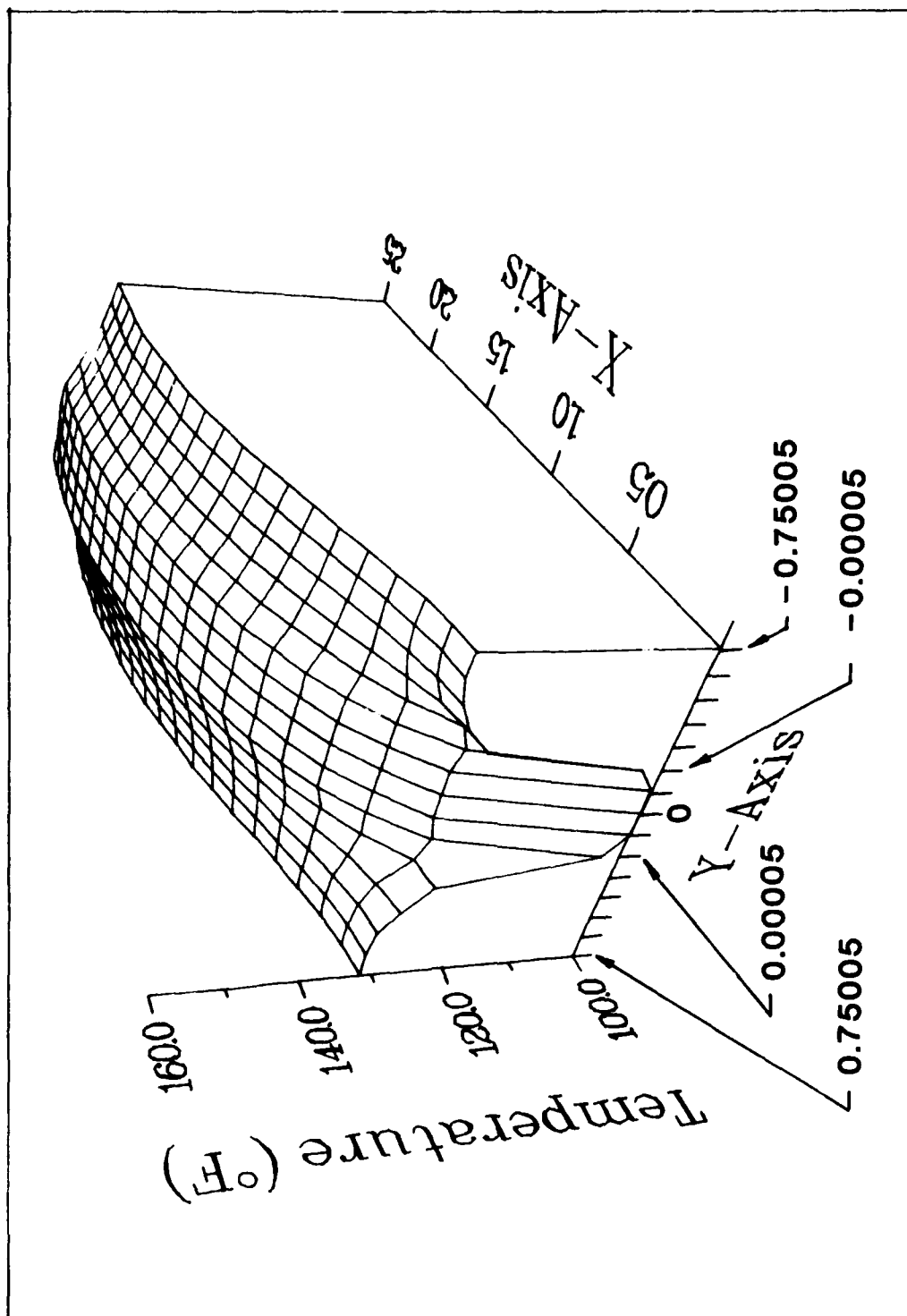


Figure 8. Temperature distributions for Case 4.

$H_2 = 0.0001$  inch,  $u_0 = 47.75$  in/sec

very much along the y-axis for better illustration in the computer plots.

The results for all four cases show that the effect of the inlet temperature (100° F is used here) is limited to a small local region around the inlet. This result shows that there exists a very thin thermal boundary layer at the inlet (i.e., the initial layer in the time-dependent problem). This results is not surprising, since the heat is generated only in the lubricant.

It is of interest to note that for all four cases, the temperature  $T$  at the second node on the boundary  $J_1$  (or  $J_2$ ) from the inlet ( $\Gamma_{2U}$ ,  $x = 0$ ) is lower than that at the adjacent node inside the solids (i.e.,  $\Omega_1$  or  $\Omega_3$ ). This means that there is a small region of local backflow of heat flux. In other words, in this region, the heat flux vector is pointing from the point in the solid to the lubricant, even though the only heat generating source in this problem is in the lubricant region.

#### CONCLUDING REMARKS

From the numerical results presented for the first model lubrication problem, we can conclude that the seemingly-unstable classical Galerkin method is uniformly stable over the range of the Peclet number from 0.01 to 100. This stability is probably a result of the subdomain of the parabolic equation being sandwiched between two adjacent subdomains which are elliptic without a convective term in the heat transfer equations. It appears that the elliptic type equations for the top and bottom subdomains play a role in stabilizing the numerical scheme even though we use the classical Galerkin method which is equivalent to the centered finite difference approximation. In the second model problem, a local backflow of the heat flux and the presence of the thermal boundary layer are illustrated. Future work should include the convective term in the bearing collar where numerical stability may not result using the present method, unless a proper weighting function is introduced

in the inner product.

#### ACKNOWLEDGMENT

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